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A TYPE OF DESIGN RELATED TO GRAPH DECOMPOSITION

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1. Introduction

In this paper we consider a decomposition of graphs. The reader is referred to [4] for any term not defined below. Let G be an undirected graph with v points and q lines without loops or multiple lines. We denote the set of all points of G by $V(G)$. G is called a regular graph of degree r if all points have the same degree r. A regular graph of degree r is called the complete graph, denoted by K_v, if $r=v-1$. A connected graph with $c+1$ points and c lines is a complete bipartite graph and is called a claw of degree c.

The decomposition of graphs plays an important role in the combinatorial theory of design of experiments. A balanced incomplete block design (BIBD) has been investigated by many authors[1,2,3,6]. In the book[5] many results are appeared. BIBD on v points with block size k and index unity may be considered as a partition of the lines of the complete graph K_v into complete subgraphs, each having k points. Since BIBD is a partition of lines, it may be called a *design with respect to line*.

In this paper we consider a *design with respect to point* like a design with respect to line. A design with respect to point is a partition of the points of graph into complete subgraphs, each having the same number of points. A necessary condition for the existence of a design with respect to point will be given. Several results will also be given.

2. Definition and a necessary condition

DEFINITION. *Let K_k be a complete graph on a set of k points. Then a graph G is said to have point-disjoint K_k-decomposition if*

$$V(G)=\bigcup_{\alpha} V(K_k{}^{(\alpha)}), \quad V(K_k{}^{(\alpha)})\cap V(K_k{}^{(\beta)})=\phi \quad (\alpha \neq \beta),$$

where $K_k{}^{(\alpha)}$ is isomorphic to K_k.

A point-disjoint K_k-decomposition of G gives a design with respect to point having block size k. The following example illustrates a decomposition.

EXAMPLE 1. A point-disjoint K_2-decomposition of the graph G in Fig. 1 is as follows:

$$\{p_1, p_2, p_3, p_4, p_5, p_6\} = V(K_2{}^{(1)})\cup V(K_2{}^{(2)})\cup V(K_2{}^{(3)}),$$

where $K_2{}^{(1)}$, $K_2{}^{(2)}$, and $K_2{}^{(3)}$ is the complete subgraphs given in Fig. 2.

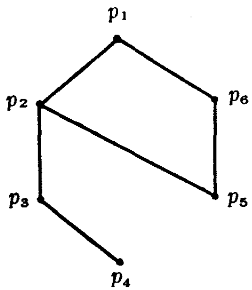

Fig. 1. Graph in Example 1

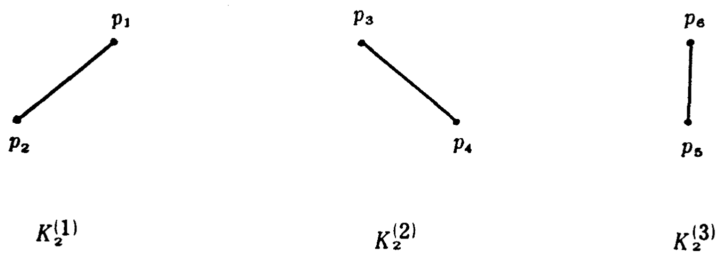

Fig. 2. Complete subgraphs of the graph in Fig. 1

Now we give a necessary condition in the following.

THEOREM 1. *Let v be the number of points of G, i.e., $|V(G)| = v$. Let d_i be the degree of the i-th point of G. Then a necessary condition that G has point-disjoint K_k-decomposition is*

(i) *k is a factor of v,*

(ii) *$d_i \geq k-1$ for each i.*

PROOF. The condition (i) is obviously necessary. Since each point is contained in a complete subgraph, say $K_k^{(\alpha)}$, and since the degree of the point in the $K_k^{(\alpha)}$ is k-1, it follows that the degree of this point in G is not less than k-1. Hence we obtain the condition (ii).

For $k=1$, since K_1 is a single point, any graph has always point-disjoint K_1-decomposition. However, when $k \geq 2$, we shall observe by an example that Conditions (i) and (ii) in Theorem 1 are not always sufficient.

EXAMPLE 2. Consider a claw G of degree 3 ($v=4$) in Fig. 3. If we consider $k=2$, then "$k=2$" satisfies Conditions (i) and (ii) in G. However, we can check easily that G does not have point-disjoint K_2-decomposition.

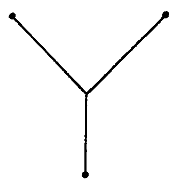

Fig. 3. Claw G of degree 3

3. Sufficient conditions

In this section we shall give two theorems.

THEOREM 2. *If the originl graph G is K_v, then G has point-disjoint K_k-decomposition for any factor k of v.*

PROOF. Put $v=\lambda k$ and partition $V(G)$ into λ subsets of k points each. Then the induced subgraph of G for each subset is obviously a complete graph K_k. Thus G has point-disjoint K_k-decomposition.

Consider a graph G' defined as follows: (1) There is a one-to-one correspondence between the points in G' and the lines in G. (2) There is a line joining two points in G' when their corresponding lines in G are incident with a common point. G' is called the line graph of G and is denoted by $L(G)$. The line graph of the graph (claw of degree 3) in Fig. 3 is, for example, K_3. We prove

THEOREM 3. *Let v be a positive integer and let c be a factor of $\binom{v}{2}$ satisfying $v \geq 2c$. Then $L(K_v)$ has point-disjoint K_c-decomposition.*

PROOF. It is shown by[7] that K_v can be decomposed into a union of line-disjoint claws of degree c each. In this decomposition we denote the set of all line-disjoint claws of degree c by $\bar{L}=\{G_1, G_2, \cdots, G_\lambda\}$, where $\lambda=\binom{v}{2}/c$. Let $E(G_\alpha)$ be the set of all lines of G_α for $\alpha=1,2,\cdots,\lambda$ We can see easily that the line graph of a claw of degree c is a complete graph K_c. It is also seen easily that $V(L(G_\alpha)) \cap V(L(G_\beta))=\phi$ holds for G_α and G_β in $\bar{L}(\alpha \neq \beta)$. Furthermore, since $\bigcup_{\alpha=1}^{\lambda} E(G_\alpha)$ is the set of all lines of the complete graph K_v, we have $\bigcup_{\alpha=1}^{\lambda} V(L(G_\alpha))=V(L(K_v))$. This completes the proof.

EXAMPLE 3. Consider K_6 on point set $\{p_1, p_2, p_3, p_4, p_5, p_6\}$ and $c=3$. Then c is a factor of $\binom{6}{2}=15$. Let $\bar{L}=\{u_{ij} \mid 1 \leq i < j \leq 6\}$, where u_{ij} is the line joining p_i and p_j. Then $\bar{L}$ can be decomposed into disjoint five subsets:

$$\bar{L}=E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5,$$

where $E_1 = \{u_{12}, u_{13}, u_{16}\}$, $E_2 = \{u_{23}, u_{24}, u_{26}\}$, $E_3 = \{u_{34}, u_{35}, u_{36}\}$, $E_4 = \{u_{14}, u_{45}, u_{46}\}$ and $E_5 = \{u_{15}, u_{25}, u_{56}\}$. Note that the graphical structure of each E_α is a claw of degree 3. The line graph of K_6 whose point set is $\bar{L}$ can be decomposed as seen in Fig. 4, where we note that each part in Fig. 4 is K_3.

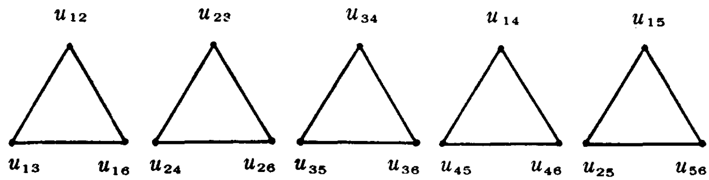

Fig. 4. Decomposition of the line graph of K_6

4. Miscellaneous results

THEOREM 4. *If a graph G has point-disjoint K_k-decomposition, then G has point-disjoint K_t-decomposition for any factor t of k.*

The proof follows immediately from Theorem 2.

In the following we consider a graph having a hamiltonian path which is defined to be an alternating sequence of points and lines that meets every point exactly once.

THEOREM 5. *Suppose the number v of points in G is even. Then if G has a hamiltonian path, G has point-disjoint K_2-decomposition.*

PROOF. Let $p_1, x_1, p_2, \cdots, p_{v-1}, x_v, p_v$, be a hamiltonian path in G, where p_i denotes the i-th point and x_i denotes the line joining p_i and p_{i+1}. Since v is even, we have

$$V(G) = V(K_2^{(1)}) \cup V(K_2^{(2)}) \cup \cdots \cup V(K_2^{(v/2)}),$$

where $V(K_2^{(i)}) = \{p_{2i-1}, p_{2i}\}$ is the set of adjacent points for $i = 1, 2, \cdots, v/2$. Thus G has point-disjoint K_2-decomposition.

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