An Observation of the Stability: Statics and Dynamics[1)]

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[1] Introduction

We are doing economic life day by day. Our purpose of economic life is to full up the satisfaction of economic wants within a limited resources. Therefore, a symmetry of economics is also individual behaviour in the production or consumption of goods and services, the exchange in the market, and the distribution for the member of the world, we could be simply summarize above to this. For analyzing to those problems we let to use some mathematical tools in modern economics. Accordingly, we are to solve above the symmetrical problems by using tools of mathematics.

Then, we know to make it an existence on the difference of Statics and Dynamics in the economic theory. However, the concept on the Statics and Dynamics is difficult to define it exactly. Nevertheless, the general idea is really important for an economic analysis. Now then, at first, let us deal with a concept of Statics and Dynamics. Next, we deal with a problem on the Stability of Statics or Dynamics, continuously the Correspondence Principle.

1) I have an imagination to take up too great problem. Rather, this title may have to change another one, but I try to research about this. Therefore, I should bring about too much mistakes literally or analytically. Moreover, I take the responsibility for all of them in this paper. Then, I expect to point upon the individual mistake out of my friendly professor.

〔2〕 A Concept of Statics and Dynamics

The economists who are representative Statics and Dynamics are considered J. R. Hicks and P. A. Samuelson.[2)] Professor Hicks describes the following about Statics and Dynamics: I call Economic Statics those parts of economic theory where we do not trouble about dating; Economic Dynamics those parts where every quantity must be dated. For example, in economic statics we think of an entrepreneur employing such-and-such quantities of factors and producing by their aid such-and-such quantities of Products; but we do not ask when the factors are employed and when the products come to be really. In economic dynamics we do ask such questions; and even pay special attention to the way changes in these dates affect the relations between factors and products.[3)]

In Hicksian consideration, he makes allowance for dating or no dating, but Professor Samuelson makes consideration of the behavioural system: Statical refers to the form and structure of the postulated laws determined the behavior of the system. An equilibrium defined as the intersection of a pair of curves would be statical. Ordinarily, it is timeless in that noting is specified concerning the duration of the process. In defining the term Dynamical, at least two possibilities suggest themselves. First, it may be defined as a general term including statical as a special rather degenerate case. Or, on the other hand, it may be defined as the totality of all systems which are not statical.[4)] We may

2) J. R. Hicks; Value and Capital. 2 nd. ed. Oxford, 1946: P. A. Samuelson Foundations of Economic Analysis. 4 th. printing. Harvard Univ. Press, 1970.

3) Hicks; op. cit. Chapter IX, The Method of Analysis. P. 115.

4) Samuelson; op. cit. Chapter XI, Some Fundamentals of Dynamical Theory. P. 313.

so that although an arbitrary initial situation deverged from this equilibrium condition, together with proceeding of time the devergence is say that a system is dynamical if its behavior over time is determined by functional equations in which variable at different point of time are involved in an essential way.[5)]

In summary, Dynamics is defined as the process of successive change of the variable from one time to next time, this model, therefore, should be demonstrated upon such process. On the one hand, Statics is defined as the process changing each datum when a various datum changes. Moreover, on the other hand, as Samuelson points out the following: As simple statical system as defined above would also have the property of being stationary.[6)] An idea of stationary state is the way of more one analysis in Statics. Stationary State is the status that when the price and wage is all constant, then the activities, for example, production, consumption, exchange, etc., are quite repeatedly repetitional with same level. In stationary state, as one datum is happened any change, this state is disturbed, but passing over time consists this new stationary state corresponding on the change of datum. We compare with such a new stationary and old stationary state, and analyze how exerts a change of datum effectiveness to the endogenous variables.[7)] However, we have not dealing with in this paper rather stability condition in the light of this title.

Then, stability is that, although any point of equilibrium deverged from real equilibrium, this motions again turn back to initial condition,

5) Samuelson; op. cit. Chapter XI. P. 314.

6) Samuelson; op. cit. Chapter XI. P. 313.

7) Imai, Uzawa, Komiya, Negishi, & Murakami; Price Theory I, Iwanami, 1971; Introduction. P. P. 22～23.

necessarily convergent to initial point of equilibrium. Professor Samuelson describes about concepts of stability[8] as follow: a) Any position of equilibrium is stable if deviations from it remain bounded. If no motions go out to infinity, then each is stable. b) Although the system may not exactly repeat itself starting from an arbitrary initial state, yet in general it will return to the vicinity of its initial state and nearly repeat its motion during a long interval of time. This is quite different from the usual meaning of stability. c) A very common use of stability in nonconservative physical system is the one which I have termed stability of the first kind. It holds when every motion approaches in the limit the position of equilibrium. There is stability of the first kind in the small if in a sufficiently small neighborhood of a given motion all motions are stable. d) All conservative physical systems are reversible in time, and volumes are preserved in phase space. This rules out stabilility of the first kind. e) There remain still other notions such as parmanent stability, semi-parmanent stability in which the above properties hold for long periods of time, complete or trigonometric stability in which the motion can be approximated by certain harmonic sums, etc. In summary this is such things as we describe above.

〔3〕 A Stability of Comparative Statics

Comparative statics is to analyze about what effect exerts a change of one datum to an initial equilibium values. At the first, let us examine on the consumer behaviour. We make usually purchase the different commodities with a given income Y, then our income consists

8) Samuelson; op. cit. chapter XI. P. P. 333～334.

of the sum of expenditure for spending it upon the respective commodities. When we buy a commodities. The combinations of goods are indicated by the level of utility which can represent the utility function.

Now, we indicate the different commodities by x_i, the prices of each commodities P_i, a given income $\overline{Y}$, and a level of utility u, then we have respectively a utility function and a equation of income.

$$u=u\left(\sum_{i=1}^{n} x_i\right) \tag{3—1}$$

$$\overline{Y}=\sum_{i=1}^{n} P_i x_i \tag{3—2}$$

where we assume that our income expends wholly to purchasing the commodities, moreover we consider that a level of utility is measurable or orderly.

The consumer is able to purchase optimum amounts of commodities with maximizing utility function subject to given income which we call it budget constraint or restraint condition. To solve these equations we use the method of the Lagrange multiplier λ. Then, writting the Lagrangean equation

$$L=u\left(\sum_{i=1}^{n} x_i\right)+\lambda\left[\overline{Y}-\left(\sum_{i=1}^{n} P_i x_i\right)\right] \tag{3—3}$$

Differentiating equation (3-3), setting the respective partial derivatives equal zero, we can derive the well-known first order equilibrium conditions.

$$\sum_{i=1}^{n} u_i-\lambda\sum_{i=1}^{n} P_i=0$$

or
$$\sum_{i=1}^{n} u_i=\lambda\sum_{i=1}^{n} P_i \tag{3—4}$$

Then, eliminating λ, we reduce to the following.

$$\frac{u_1}{P_1}=\frac{u_2}{P_2}=\cdots\cdots=\frac{u_{n-1}}{P_{n-1}}=\frac{u_n}{P_n}$$

or $$\frac{u_1}{u_n}=\frac{P_1}{P_n},\quad \frac{u_2}{u_n}=\frac{P_2}{P_n},\quad \cdots\cdots,\quad \frac{u_{n-1}}{u_n}=\frac{P_{n-1}}{P_n} \qquad (3-5)$$

These equations state that in equilibrium the marginal rate of substitution between any two goods must equal the ratio of their money prices. Stated somewhat differently these equations mean that the marginal rate of substitution between any good and the numéraire must equal its price in terms of the numéraire. These $n-1$ equations together with equation (3-2) are sufficient to determine the n known quantities x_1, x_2, ..., x_n in the individual's equilibrium collection of commodities.[9]

In order that u should be a true maximum we must have not only $du=0$ but also $d^2u<0$. Then, expanding equation (3-1), we are able to obtain the conditions of maximum.

$$du=\sum_{i=1}^{n} u_i dx_i \qquad (3-6)$$

$$d^2u=\sum_{i=1}^{n}\sum_{j=1}^{n} u_{ij}dx_i dx_j \qquad (3-7)$$

where dx_i is indicated the first partial derivative, and u_{ij} for the second partial derivative. Cousequently the conditions for $d^2u<0$ for all values of dx_1, dx_2, ..., dx_n, such that $du=0$, are that the determinants

$$\begin{vmatrix} 0 & u_1 & u_2 \\ u_1 & u_{11} & u_{12} \\ u_2 & u_{12} & u_{22} \end{vmatrix},\quad \begin{vmatrix} 0 & u_1 & u_2 & u_3 \\ u_1 & u_{11} & u_{12} & u_{13} \\ u_2 & u_{12} & u_{22} & u_{23} \\ u_3 & u_{13} & u_{23} & u_{33} \end{vmatrix},\quad \cdots\cdots,\quad \begin{vmatrix} 0 & u_1 & u_2 \cdots u_n \\ u_1 & u_{11} & u_{12}\cdots u_{1n} \\ u_2 & u_{12} & u_{22}\cdots u_{2n} \\ \cdots & \cdots & \cdots\cdots \\ u_n & u_{1n} & {}_{2n}\cdots u_{nn} \end{vmatrix} \qquad (3-8)$$

9) J. L. Mosak ; General Equilibrium Theory in International Trade. Principia Press, 1944, P. 12. Notice that in the usual case this is described generally that the relative rate of marginal utility, hence equals the marginal rate of substitution between any two goods, equals the rate of relative price. The result is also equivalent to either Mosak's case or usual case.

should be alternatiuely positive and negative.[10] In this series the last determinant is U, the next to last is the cofactor U_{nn} of u_{nn} in U, the one preceding that is the cofacter $U_{nn,(n-1)(n-1)}$ of $u_{nn,(n-1)(n-1)}$ in U_{nn}, et cetera. Consequently the stability conditions state that

$$\frac{U_{nn}}{U},\quad \frac{U_{nn,(n-1)(n-1)}}{U},\quad \cdots\cdots,\quad \frac{U_{nn,(n-1)(n-1)\cdots 22}}{U} \qquad (3-9)$$

are alternatively negative and positive. Since the order of numbering the commodities is arbitrary, these conditions are equivalent to the conditions that

$$\frac{U_{11}}{U},\quad \frac{U_{11,22}}{U},\quad \frac{U_{11,22,33}}{U},\quad \cdots\cdots,\quad \frac{U_{11,22,33,\cdots\cdots,(n-1)(n-1)}}{U} \qquad (3-10)$$

shall be alternately negative and positive.[11] Recalling that $x_{ij}=U_{ij}/U$, we obtain the condition that the determinants involving the substitution terms,

$$x_{11},\ \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix},\ \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix},\ \cdots,\ \begin{vmatrix} x_{11} & x_{12} & \cdots\cdots x_{1,n-1} \\ x_{21} & x_{22} & \cdots\cdots x_{2,n-1} \\ \cdots & \cdots & \cdots \\ x_{n-1,1} & x_{n-1,2} & \cdots x_{(n-1)(n-1)} \end{vmatrix} \qquad (3-11)$$

shall be alternately negative and positive.[12]

Next, we should examine about the behaviour of firm. The firm may be thought of as employing less factors to produce more products. we assume that they begin to produce quantities of products y_1, y_2, …, y_m as employing various quantities of factors v_{m+1}, v_{m+2}, …, v_n. In such a case the production function is indicated as follows.

$$f\Big(\sum_{i=1}^{m} y_i\,;\ \sum_{j=m+1}^{n} v_j\Big)\Big)=0, \qquad (3-12)$$

The object of entrepreneur is to maximize its profit upon this con-

10) Hicks; op. cit., Mathematical Appendix. P. 306.

11) Mosak; op. cit., Chapter I. P. 13.

12) Mosak; op. cit., Chapter I. P. 24. See; footnote 9.

straint conditions. Therefore, indicating the profit by R

$$R=\sum_{i=1}^{m} P_i y_i - \sum_{j=m+1}^{n} q_j v_j \qquad (3-13)$$

where p_i and q_j is represent the prices of products or factors respectively. Assuming perfect competition, we can solve its problem by introducing Lagrange multiplier μ. Accordingly, a Lagrangean equation L is indicated the following like an utility function.

$$L=\sum_{i=1}^{m} P_i y_i - \sum_{j=m+1}^{n} q_j v_j + \mu\left[f\left(\sum_{i=1}^{m} y_i\,;\ \sum_{j=m+1}^{n} v_j\right)\right] \qquad (3-14)$$

The first order conditions are retained out of equation (3-14).

$$\sum_{k=1}^{n} P_k - \mu \sum_{k=1}^{n} f_k = 0 \qquad (3-15)$$

It μ is eliminated, this gives us $n-1$ equations, which, together with the production function, determine the n quantities $y_1, y_2, \dots, y_n$, because we are able to consider it which are treated the factors as negative products.

Whence the condition for maximum are

$$d(R-\mu f)=0, \qquad d^2(R-\mu f)=0 \qquad (3-16)$$

Then, since R is linear, $d^2R=0$; the second condition therefore implies that $d^2f>0$, subject to $df=0$. Expanding, when we derive a similar set of stability conditions, then the determinants

$$\begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{12} & f_{22} \end{vmatrix}, \quad \begin{vmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & f_{11} & f_{12} & f_{13} \\ f_2 & f_{12} & f_{22} & f_{23} \\ f_3 & f_{13} & f_{23} & f_{33} \end{vmatrix}, \quad \cdots, \quad \begin{vmatrix} 0 & f_1 & f_2 & \cdots f_n \\ f_1 & f_{11} & f_{12} & \cdots f_{1n} \\ f_2 & f_{12} & f_{22} & \cdots f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_n & f_{1n} & f_{2n} & \cdots f_{nn} \end{vmatrix} \qquad (3-17)$$

must all be negtive.[13)]

13) Hicks; op. cit., Mathematical Appendix. P. 320.

[4] Thc Statical Stability of Multi-exchange Equilibrium

In general the various quantities of n goods are exchanged by n individuals bringing to the market them under conditions of perfect competition. The equilibrium of exchange requires that the total quantity demanded of all commodities equate to the total quantity supplied of all goods.

$$X_i = Y_i \qquad (i=1, 2, \cdots, n) \tag{4—1}$$

Discribed above, we shall indicate that X_n is the quantities bought for all individuals x_i, Y_n the amounts brought by all individuals y_i, and the prices of commodities p_i. If the system is in equilibrium, then the demand for all commodities must equal the supply of all goods.

$$\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i y_i \tag{4—2}$$

Summing for all individuals, we have

$$\sum_{i=1}^{n-1} p_i X_i = \sum_{i=1}^{n-1} p_i Y_i \tag{4—3}$$

When we consider that one good x_n has to be taken in terms of numéraire, therefore any one price p_n equals unity, then the remaining prices $\sum_{i=1}^{n-1} p_i$ have to be determined. Since the equations (4-3) must necessarily hold, namely regardless of whether (4-1) satisfied or not, it follows that if $n-1$ equations are satisfied, the nth equation must be satisfied too, therefore, there are only $n-1$ equations to determine the $n-1$ prices.[14)]

Thus, we have the $n-1$ unknowns to be determined by the $n-1$ equations (4-1). Notice that the supply of commodities Y_i is considered to be able to taken as given. Therefore, we reduce the equili-

14) Hicks; op. cit., Mathematical Appendix. P. 314.

brium system to the $n-1$ dimensional.

$$\sum_{i=1}^{n-1} X_i(q_1, q_2, \cdots, q_{n-1}) = \sum_{i=1}^{n-1} \overline{Y}_i \qquad (4-4)$$

where q_i is the $n-1$ price ratios. This system provides us with a set of $n-1$ independent equations to determine the $n-1$ price ratios under a given set of conditions. The given set of conditions are:

(1) the preference function for each individnal, and

(2) the initial collection of goods for each individual.

A change in these givens will change the system (4-4) and given rise to a new set of equilibrium price ratios.[15)]

Now, in the system (4-4) we have the $n-1$ unknowns to be determined in equilibrium and the price ratios q_i. Then, we must consider the effects of a change in a price ratio. Professor Hicks gives us the conditions for the stability of multi-exchange equilibrium as follows. Since $\overline{Y}_i$ can be taken as constant, the conditions for the stability of exchange can be got by examining the sign of $\frac{dX_i}{dp_i}$. In order for equilibrium to be perfectly stable, $\frac{dX_i}{dp_i}$ must be negative:

(1) when all other prices are unchanged;

(2) when p_j is adjusted so as to maintain equilibrium in the market for x_j, but all other prices are unchanged;

(3) when p_j and p_k are similarly adjusted;

and so on, until we have adjusted all prices, excepting p_n (which is necessarily unity).[16)]

Then, we try to consider the conditions for stability, hence differentiating the equations (4-1) with respect to p_i, we have the following.

15) Mosak; op. cit., Chapter II. P. 39.

16) Hicks; op. cit., Mathematical Appendix. P. 315.

$$\frac{\partial X_i}{\partial P_i}=\frac{\partial X_i}{\partial P_i}+\frac{\partial X_i}{\partial P_j}\,\frac{dP_j}{dP_i}+\frac{\partial X_i}{\partial P_k}\,\frac{dP_k}{dP_i}$$

$$0\ =\frac{\partial X_j}{\partial P_i}+\frac{\partial X_j}{\partial P_j}\,\frac{dP_j}{dP_i}+\frac{\partial X_j}{\partial P_k}\,\frac{dP_k}{dP_i} \qquad (4-5)$$

$$0\ =\frac{\partial X_k}{\partial P_i}+\frac{\partial X_k}{\partial P_j}\,\frac{dP_j}{dP_i}+\frac{\partial X_k}{\partial P_k}\,\frac{dP_k}{dP_i}$$

Eliminating $\frac{dp_j}{dp_i}$ and $\frac{dp_k}{dp_i}$,

$$\frac{dX_i}{dP_i}=\begin{vmatrix}\frac{\partial X_i}{\partial P_i} & \frac{\partial X_i}{\partial P_j} & \frac{\partial X_i}{\partial P_k}\\ \frac{\partial X_j}{\partial P_i} & \frac{\partial X_j}{\partial P_j} & \frac{\partial X_j}{\partial P_k}\\ \frac{\partial X_k}{\partial P_i} & \frac{\partial X_k}{\partial P_j} & \frac{\partial X_k}{\partial P_k}\end{vmatrix} : \begin{vmatrix}\frac{\partial X_j}{\partial P_j} & \frac{\partial X_j}{\partial P_k}\\ \frac{\partial X_k}{\partial P_j} & \frac{\partial X_k}{\partial P_k}\end{vmatrix} \qquad (4-6)$$

To be perfectly stable, $\frac{dX_i}{dp_i}$ must be negative. Taking all similar condition together, and remembering that they must hold for the market in every x_i, the stability conditions is emerged by the Jacobian determinants.

$$\frac{\partial X_i}{\partial P_i},\quad \begin{vmatrix}\frac{\partial X_i}{\partial P_i} & \frac{\partial X_i}{\partial P_j}\\ \frac{\partial X_j}{\partial P_i} & \frac{\partial X_j}{\partial P_j}\end{vmatrix},\quad \begin{vmatrix}\frac{\partial X_i}{\partial P_i} & \frac{\partial X_i}{\partial P_j} & \frac{\partial X_i}{\partial P_k}\\ \frac{\partial X_j}{\partial P_i} & \frac{\partial X_j}{\partial P_j} & \frac{\partial X_j}{\partial P_k}\\ \frac{\partial X_k}{\partial P_i} & \frac{\partial X_k}{\partial P_j} & \frac{\partial X_k}{\partial P_k}\end{vmatrix} \qquad (4-7)$$

to be alternatively negative and positive.[17)]

In order for this stability conditiors to clearly understand, let us consider with respect to a change in q_1. Differentiating the equations (4-4), we have the system.

$$\frac{d(X_1-\overline{Y}_1)}{dq_1}=\frac{dX_1}{dq_1}=\sum_{i=1}^{n-1}\frac{\partial X_1}{\partial q_i}\,\frac{dq_i}{dq_1}$$

17) Hicks, op. cit., Mathematical Appendix. P. 315.

$$\frac{d(X_2-\overline{Y}_2)}{dq_1}=\frac{dX_2}{dq_1}=\sum_{i=1}^{n-1}\frac{\partial X_2}{\partial q_i}\,\frac{dq_i}{dq_1} \qquad (4-8)$$

$$\frac{d(X_k-\overline{Y}_k)}{dq_1}=\frac{dX_k}{dq_1}=\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}\frac{\partial X_k}{\partial q_i}\,\frac{dq_i}{dq_1}$$

Let us assume,

$$a_{ij}=\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\frac{\partial X_i}{\partial q_j}=P_n\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\frac{\partial X_i}{\partial q_j} \qquad (4-9)$$

Further, let us define Jacobian determinants,

$$J\equiv\begin{vmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1,n-1} \\ a_{21} & a_{22} & \cdots\cdots & a_{2,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-1,1} & a_{n-1,2} & \cdots\cdots & a_{n-1,n-1} \end{vmatrix} \qquad (4-10)$$

and moreover assuming J_{ij} the cofactor of a_{ij} in J, it follows from the equations (4-8) that

$$\frac{J}{J_{11}}=\frac{dX_1}{dq_1}=P_n\frac{dX_1}{dP_1}<0 \qquad (4-11)$$

If the market is to be perfectly stable, then $\frac{dX_1}{dq_1}$ must be negative regardless of whether the other price ratios remain constant or are adjusted so as to maintain equilibrium in those markets. For each additional price that is kept constant we have one less unknown and one less equation to set equal to zero in the equations (4-8). Solving for every such a system we obtain as the conditions of perfect stability that the determinants

$$a_{11},\ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},\ \cdots\cdots,\ \begin{vmatrix} a_{11} & a_{12} & \cdots\cdots a_{1,n-1} \\ a_{21} & a_{22} & \cdots\cdots a_{2,n-1} \\ a_{n-1,1} & a_{2,n-1} & \cdots\cdots a_{n-1,n-1} \end{vmatrix} \qquad (4-12)$$

shall be alternatively negative and positive.[18]

18) Mosak, op. cit., Chapter II. P. 41.

Professor Allen describes the following criticism[19] about things analyzed above: The Hicksian notion of perfect stability can be criticised on the ground that it is defined by analogy with the case of a single market. Only in the simple case is a particular dynamic model constructed as a criterion for stability. The general case of many markets is not considered, at least explicitly, in the light of a dynamic model showing the movements of the inter-related prices over a time. The analysis is clearly incomplete. An unsuspected difficulty will then emerge from the fact that a_{rs} and a_{sr} are generally different, since it cannot be assumed eigher that $\frac{\partial X_r}{\partial P_s}=\frac{\partial X_s}{\partial P_r}$ or that $\frac{\partial Y_r}{\partial P_s}=\frac{\partial Y_s}{\partial P_r}$.

However we shall be enough to understand the conditions for equilibrium stability even in a single market, and in general bring forward the symmetric determinants in the static model.

〔5〕 The Stability of Walrasian General Equilibium

As point out Professer Samuelson[20]: For a simgle market, according to Professor Hicks, equilibrium is stable if an increase in demand raises price. For multiple markets equilibrium is imperfectly stable if an increase in demand for a simgle good its price after all other prices have adjusted themselves; the equilibrium is perfectly stable if an increased demand for a good raises its price even when any subset of other prices is arbitrarily held constant.

To test these criteria we take up the Walrasian form, namely in the general equilibrium analysis the demand and supply functions for goods are fundamentally functions of all goods.

19) R. G. D. Allen; Mathematical Economics. 2 nd. Macmillan, 1959. Chapter 10, General Economic Equilibrium. P. P. 328～329.

20) Samuelson; op. cit., Chapter IX. P. 270.

$$E_i = x_i(P_1,P_2,\cdots,P_n) - y_i(P_1,P_2,\cdots,P_n) = E_i(P_1,P_2,\cdots,P_n)$$
$$[i=1,2,\cdots,n] \qquad (5-1)$$

where E_i represent static excess demand for the i th good, x_i the demand for the i th good, and y_i the supply of the i th good. The equilibrium points for each good is the conditions that all the excess demands are zero. Moreover, if the excess demand is positive, then the price of a good increases, and if the excess demand is negative, then the price decreases. The statical stability conditions of partial equilibrium is therefore that a change in price becomes cause a change to the opposite direction in the excess demand, so that we have

$$\frac{dE_i}{dP_i} < 0 \qquad i=1,2,\cdots,n \qquad (5-2)$$

Professor Gandolfo describes as follows[21]: Of course, account must be taken of the fact that the change in a price influences not only its own excess demand, but also, in principle, all the other excess demands. Thus conditions (5 — 2) must be qualified. Hicks' qualifications consist in the distinction between imperfect and perfect stability. Stability is imperfect when (5-2) holds only when, giving a change in the j th price, all the other prices have adjusted in such a way that all the other markets are again in equilibrium. Stability is instead perfect when (5-2) holds in any case, that is when

(1) all the other prices have adjusted as in the previous case;

(2) all the other prices have remained constant;

(3) any subset of k other prices have varied so that equilibrium in the respective markets has been restored, where as the remaining $m-k$ prices have remained constant.

21) Giancarlo Gandolfo; Mathematical Methods and Models in Economic Dynamics. North-Holland, 1971. Part II, Chapter 9. P.P. 275～276.

Then, in order to find these conditions, we should totally differentiate the equations (5-1).

$$dE_i=\frac{\partial E_i}{\partial P_1}dP_1+\frac{\partial E_i}{\partial P_2}dP_2+\cdots+\frac{\partial E_i}{\partial P_n}dP_n \qquad i=1,2,\cdots,n \qquad (5-3)$$

Rewritting the equations (5-3) to the determinant system;

$$dE_i=\begin{pmatrix} a_{11} & a_{12}\cdots\cdots a_{1n} \\ a_{21} & a_{22}\cdots\cdots a_{2n} \\ \cdots\cdots & \cdots\cdots \\ a_{n1} & a_{n2}\cdots\cdots a_{nn} \end{pmatrix} \begin{pmatrix} dP_1 \\ dP_2 \\ \vdots \\ dP_n \end{pmatrix} \qquad (5-4)$$

where $a_{ij}\equiv\frac{\partial E_i}{\partial P_j}$.

Now, we assume that when the price of the j th good has changed, then all the other prices varied as to be again in equilibrium, since the equilibrium has been restored. In this case we have the following.

$$dE_j=a_{j1}dP_1+a_{j2}dP_2+\cdots+a_{jn}dP_n=0 \qquad (j=1,2,\cdots,n) \qquad (5-5)$$

Solving with respect to dp_i we obtain

$$dP_j=dE_j\frac{D_{jj}}{D} \qquad (5-6)$$

where D is the Jacobean determinant, D_{jj} the cofactor of a_{jj}[22] in D. From the equation (5-6) we have

$$\frac{dE_j}{dP_j}=\frac{D}{D_{jj}} \qquad (5-7)$$

If the equation (5-2) is satisfied, then D and D_{jj} must be of opposite sign. Therefore, it follows that the necessary and sufficient conditions for imperfect stability are that all the principal minors of order $n-1$ of D have a sign opposite to sign of D.[23]

22) We cited from Gandolfo's work, op. cit., P. 277. Note that, since the sum of the subscripts of a_{jj} is even, the factor is the same as the minor; moreover, this minor, by its very deffinition, is a principle minor of order $n-1$.

23) Gandolfo; op. cit., P. 227.

We consider the perfect stability in which, the price of the j th good having varied, other price adjusts in such a way that the equilibrium is restored in the h th market, while all the another prices is remained constant. In this case, we have the following equations.

$$\begin{aligned} dE_j &= a_{jj}dP_j + a_{jh}dP_h \\ dE_h &= a_{hj}dP_j + a_{hh}dP_h = 0 \end{aligned} \qquad (5-8)$$

Solving for dp_j, we have

$$dP_j = dE_j \frac{a_{hh}}{\begin{vmatrix} a_{jj} & a_{jh} \\ a_{hj} & a_{hh} \end{vmatrix}} \qquad (5-9)$$

Thus, $\frac{dE_j}{dP_j}$ is negative if and only if $\begin{vmatrix} a_{jj} & a_{jh} \\ a_{hj} & a_{hh} \end{vmatrix} > 0$, since a_{hh} is negative as discribed above. So, it follows that $a_{jj} < 0$ for all j is the first condition for perfect stability, the denominator of equation (5-9) is the second minor of D, and the second minor of D must be positive.

Examining continuonsly adjustments in three, four (markets) and so on, we can obtain the necessary and sufficient conditions for perfect stability. Employing in such a way that we caught the solution, we retain all the minors of r of D having the sign of $(-1)^r$. This is that the principal minors must alternatively have the sign starting from minus, namely

$$a_{jj} < 0, \quad \begin{vmatrix} a_{jj} & a_{hj} \\ a_{hj} & a_{hh} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{jj} & a_{hj} & a_{hj} \\ a_{hj} & a_{hh} & a_{kh} \\ a_{kj} & a_{kh} & a_{kk} \end{vmatrix} < 0, \quad \cdots j \neq h \neq k \quad (5-10)$$

This is the same as saying that it is equivalent to the Hicks conditions of static stability.

〔6〕 The Dynamic Stability of Walrasian Equilibium

So far, we reserched about the stability conditions in the static system, however we turn here to examine the dynamic system. Using the assumptions indicated above, the dynamic system represents the following differential equation.

$$\dot{P}_i = k_i E_i (P_1, P_2, \cdots, P_n) \qquad i=1,2,\cdots,n \tag{6—1}$$

where $\dot{P}_i \equiv \frac{dP_i}{dt}$ and k_i is the speed of adjustment, and E_i the excess demand. Of course, these equations (6-1) state that if the excess demand for any good is positive, then its price must rise, and vice versa.

Expanding the right side in the equations (6-1) around the equilibrium prices, we have

$$\frac{d(P_i - P_i{}^0)}{dt} = k_i a_{i1}(P_1 - P_1{}^0) + k_i a_{in}(P_n - P_n{}^0) \quad i=1,2,\cdots,n \tag{6—2}$$

where $p_i{}^0$ is the equilibrium value of the i th price. For simplification, we assume that k_i is unity for all i, moreover indicating $(p_i - p_i{}^0)$ by $\bar{p}_i$, we can be written as follows.

$$\begin{aligned} \dot{\bar{P}}_1 &= a_{11}\bar{P}_1 + a_{12}\bar{P}_2 + \cdots + a_{1n}\bar{P}_n \\ \dot{\bar{P}}_2 &= a_{21}\bar{P}_1 + a_{22}\bar{P}_2 + \cdots + a_{2n}\bar{P}_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{\bar{P}}_n &= a_{n1}\bar{P}_1 + a_{n2}\bar{P}_2 + \cdots + a_{nn}\bar{P}_n \end{aligned} \tag{6—3}$$

From the theory of linear differential equations, if $p_i \rightarrow p_i{}^0$ as $t \rightarrow \infty$, then the system is stable, so that the conditions for the dynamic stability of equilibrium are indicated the following.

$$\lim_{t \to \infty} P_i(t) = 0 \tag{6—4}$$

The roots of the characteristic equation of system (6-3) are negative if real parts. The characteristic equation of the system is

$$\Phi(\lambda)=\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots a_{nn}-\lambda \end{vmatrix} = |A-\lambda \mathrm{I}| = |a_{ij}-\lambda\delta_{ij}|=0 \qquad (6-5)$$

where A is a square matrix, λ a scaler variable, and I the unit matrix.[24)]

So as to pointed out Professor Samuelson[25)]: In the symmetrical case, *i.e.*, $a_{ij}=a_{ji}$, all the roots are necessarily real. If the equilibrium is to be stable, they must all be negative. This is possible if and only if A is the matrix of a negative definite quadratic form; *i.e.*, only if all principal minors alternate in sign as follows.

$$|a_{ii}|<0, \quad \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}>0, \quad \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix}<0, \quad \cdots i\neq j\neq k \quad (6-6)$$

Professor Samuelson describes[26)] that the stability criteria of Professor Hicks are seen to be correct theorems. However, where perfect symmetry is not present, the Hicks criteria are not at all necessary conditions and in many cases not sufficient. Perfect stability, like imperfect stability, is neither a necessary nor sufficient. The Hicksian conditions

24) See; Allen, op. cit.

For example, let us construct a matrix $[A-\lambda I]$. Thus

$$A-\lambda \mathrm{I}=\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}-\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix}$$

Expanding the determinant

$$f(\lambda)=|A-\lambda \mathrm{I}|=\lambda^2-4\lambda-5$$

Therefore $f(\lambda)=(\lambda+1)(\lambda-5)=0$

Thus, $\lambda=-1$, $\lambda=5$, satisfy this characteristic equation. This is the characteristic roots of matrix A.

25) Samuelson; op. cit., Chapter IX. P. 271.

26) Samuelson; op. cit., Chpter IX. P.P. 272~273.

are neccessary but not sufficient if the system is to be stable for all positive rates of adjustment in different markets; and if all off-diagonal terms are non-negative, the Hicks conditious are both necessary and sufficient for stability.

〔7〕 Keynesian System and Correspondence Principle

An equilibrium point which exists in principle, but which cannot be approached, and which is such that slightest disturbance starts a movement away from it is obviously not very relevant from an economic point of view. It turns out that the study of the dynamic stability of equilibrium is often important also in obtaining determinate comparative statics results.[27)]

It must be stressed that comparative statics does not say anything about the time path of the variables from the initial to the final equilibrium point; nor can it say whether the new equilibrium point will actually be approached. The answer to these other questions does not pertain to comparative statics but to dynamics, and precisely to that branch of dynamics which deals with the stability of equilibrium. However, the connection between comparative statics and dynamics is expressed by the principle which Professor Samuelson has called the correspondence priniciple.[28)] This correspondence principle can be a useful tool in many instances, while it makes not a panacea.

Now, let us examine it with applying the analysis of Keynesian system. The three fundamental relations in the well-known Keynesian system are indicated by the following model of aggregate demand.

27) Gandolfo; op. cit., Appendix I. P. 350.

28) Gandolfo; op. cit., Appendix I. P. 342.

$$
\begin{aligned}
Y &= C(Y,r) + I + \alpha \\
I &= I(Y,r) + \beta \qquad (7-1) \\
M &= L(Y,r)
\end{aligned}
$$

where Y stands for income, C consumption, I investment, M money supply, L demanded money, and r the interest rate. The consumption function represents relating to income and the rate of interest, investment relating to the level of income and the interest rate, and the existing amount of money relating to the level of income and the interest rate, so that respectively the consumption function, the marginal efficiency of capital schedule, and the schedule of liquidity preference. Moreover, α presents a parameter bringing about an upward shift in the propensity to consume schedule, β an upward shift in the marginal efficiency schedule, and M is taken as a parameter shifting upward the schedule of liquidity preference.

Then, we have the three unknowns (Y, r, and I) and three parameters (α, β, and M). In order that we retain the effects of these parameters with respect to the unknowns, differentiating totally the equations (7-1), we have the following.

$$
\begin{pmatrix} C_Y-1 & C_r & 1 \\ I_Y & I_r & -1 \\ L_Y & L_r & 0 \end{pmatrix}
\begin{pmatrix} dY \\ dr \\ dI \end{pmatrix}
=
\begin{pmatrix} -d\beta \\ -d\beta \\ dM \end{pmatrix}
\qquad (7-2)
$$

where the subscript is the partial derivative relating to the endogenous variable.

Using Cramer's rule to solve the three equations for the three unknowns, we find

$$
\begin{aligned}
dY &= \frac{(-d\alpha - d\beta)L_r - (I_r + C_r)dM}{\triangle} \\
dr &= \frac{(d\alpha + d\beta)L_Y + (I_Y + C_Y - 1)dM}{\triangle} \qquad (7-3)
\end{aligned}
$$

$$dM=\frac{(C_Y L_r - L_r - L_Y C_r)d\beta - (I_Y L_r - L_Y I_r)d\alpha + (I_r C_Y - I_r - I_Y C_r)dM}{\triangle}$$

where $\triangle \equiv (I_Y + C_Y - 1)L_r - (I_r - C_r)L_Y$

To obtain the partial derivatives of the unknowns with respect to the parameters, we divide both sides of equations (7-3) by the parameters. The results is the following.

$$\begin{aligned}
\frac{dY}{d\alpha} &= -\frac{L_r}{\triangle} \\
\frac{dY}{d\beta} &= -\frac{L_r}{\triangle} \\
\frac{dY}{dM} &= \frac{-I_r + C_r}{\triangle} \\
\frac{dr}{d\alpha} &= \frac{L_Y}{\triangle} \\
\frac{dr}{d\beta} &= \frac{L_r}{\triangle} \\
\frac{dr}{dM} &= \frac{I_Y + C_Y - 1}{\triangle} \\
\frac{dI}{d\alpha} &= \frac{-I_Y L_r + L_Y I_r}{\triangle} \\
\frac{dI}{d\beta} &= \frac{C_Y L_r - L_r - L_Y C_r}{\triangle} \\
\frac{dI}{dM} &= \frac{I_r C_Y - I_r - I_Y C_r}{\triangle}
\end{aligned} \tag{7—4}$$

We conventionally establish that the following assumptions are usually made.

$$C_Y > 0,\ I_Y > 0,\ L_Y > 0,\ I_r < 0,\ L_r < 0,\ C_r \gtreqless 0 \tag{7—5}$$

Then, we must determine the signs of all numerators and the denominators, Δ, in order to evaluate the equations (7-4). If we assume usually to conduct the rational consumption behaviour, then it will follow that the higher the rate of interest, the lower the consumption. Rather it must be saying that it is exceptional behaviour to the higher the interest rate, the higher the consumption. Hence, if we try to make the rational of the behaviour assumption, then the common denominator of the equations (7-4) have to be negative; *i.e.*, $\Delta < 0$.

To examine the stability of the equilibrium in our model, we must proceed to a consideration of a dynamic system. According to Professor Samuelson,[29] we assume that the marginal efficiency and liquidity preference work out in so a short time that they can be regarded as holding instantaneously, that I now represent intended investment, and this magnitude equals saving-investment only in equilibrium. If consumption should suddenly increase, national income not having a chance to change, actual saving-investment would fall short of intended saving-investment because of inventory reduction. Income would tend to rise. Similarly an excess of actual saving-investment over intended saving-investment would tend to make income fall. Consequently, the rate of change of income is proportional to the difference between intended saving-investment and actual saving-investment.

$$\begin{aligned} \dot{Y} &= I - [Y - C(Y,r) - \alpha] \\ 0 &= I(Y,r) - I + \beta \\ 0 &= L(Y,r) - M \end{aligned} \qquad (7-6)$$

where $\dot{Y} \equiv \frac{dY}{dt}$. Jacobean determinant is rewritting as follows.

$$f(\lambda) = \begin{vmatrix} C_r & C_Y - 1 - \lambda & 1 \\ I_r & I_Y & -1 \\ L_r & L_Y & 0 \end{vmatrix} = \triangle \lambda L_r = 0 \qquad (7-7)$$

Therefore, the equilibrium is stable only if $\lambda = -\frac{\triangle}{L_r}$. Since $L_r < 0$, to be stable Δ must be negative. This is all the essence of the correspondence principle.

29) Samuelson; op. cit., Chapter IX. P.P. 278～279.

〔8〕 Summary

In respect of a concept of statics and dynamics describes the following. Professor Hicks defines that economic statics is those parts of economic theory where we do not trouble about dating; economic dynamics is those parts where every quantity must be dated. Professor Samuelson defines that statics is the process changing each datum when a various datum changes; dynamics is the process of successive change of the variable from one time to next time. After all, I suppose so as to be saying the same things about the static and dynamic definitions either Professor Hicks or Professor Samuelson. For the time being, however, we have not exactly those definitions.

In the analysis of the Walrasian general equilibrium, as Professor Hicks distinguished between imperfect and perfect stability, we also dealt with so a method as him. Imperfect stability is when $\frac{dE_i}{dp_i} > 0$ holds only if, giving a change in the i th price, all the other prices have adjusted in such a way that all the other markets are again in equilibrium. Perfect stability is when $\frac{dE_i}{dp_i} < 0$ holds in any case, that is when (1) all the other prices have adjusted, (2) all the other prices remain constant, and (3) any subset of k other prices have varied so that equilibrium in the respective markets has been restored, where as the remaining $m-k$ prices have remained constant.

Comparative statics does not say anything about the time path of the variables from the initial to the final equilibrium point, nor can it say whether the new equilibrium point will actually be approached. The answer to these other quations does not pertain to comparative statics but to dynamics. However, it may be possible to obtain some information about comparative statics from dynamic considerations. Such dynamic considerations are essential of the correspondence principle, however this is not a panacea.

The analysis which we dealt with is able to apply it in the international system. In fact, these applications are seen in any parts of the international economics. For example, we can take up in the Metzleric system[30]: (1) a shift of demand from saving onto the goods of country k raises the income of country k and raises income in every other country; it cannot be provide that income in any other country rises by less than income in country k; (2) a shift of demand from country i to country k raises income in country k and lowers income in country i; it cannot be proved that income in other countries rise by less than in country k or fall by less than in country i; (3) a shift of demand from all other countries fall, or rise by less than income in country k. Or, this analysis is used the problems expanding to the international currencies and currency areas. (September 16. 1980)

30) R.A. Mundell; International Economics. 1968. Chapter 7. P.P. 100～107. Would you like to refer many paper announced around in our world.

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