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Note

Japanese Temple Geometry Problems and Inversion

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1. Introduction

In ancient Japan mathematicians used to carry their best theorems to a shrine or temple, and hang them somewhere on the wall. Such theorems were usually painted on piece of wooden board. If there was a geometric construction, then quite often the picture was very colorful. Sometimes such a picture contained decorations with flowers, plants, mountains, etc. Some of them were real pieces of art. It was probably a way to thank the Gods for the moment of enlightenment while solving the problem.

They were simultaneously works of art, religious offerings, and works of mathematics. They are called *sangaku*, which simply means mathematical tablet. These were hung in Buddhist temples and Shinto shrines throughout Japan, and for that reason the entire collection of *sangaku* problems has come to be known as “temple geometry”.

The year 1600 is the time when we should start looking for the origins of *wasan*, Japanese mathematics of the Edo period. At that time, Japan was controlled by the daimyo, or in western terms warlords, who were still fighting for dominancy. Some of them were very powerful and the country was in a constant state of unrest.

Figure 1　Sangaku of the Takemizuke shrine, Nagano perfecture

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In 1600, during the famous Sekigahara battle, the daimyo were defeated by Tokugawa Ieyasu. Three years later, Tokugawa Ieyasu became the shogun of Japan. This was the starting point of a new period in the history of Japan – a period of almost 250 years without war known as “Great Peace”. After the battle, Tokugawa Ieyasu moved to a small, at that time, provincial town of Edo, today’s Tokyo. For this reason, the rule of the Tokugawa is also known as the Edo period. The country was united and many changes started taking place.

This was also the time when the Spanish, Portuguese and Dutch tried to settle down in Japan, and strengthen their trade. At the same time, missionaries from these countries wanted to convert as many souls as possible. The trade with foreigners was not considered problematic, however, conversion of people to Christianity was not very welcomed by the two main religions in Japan – Shinto and Buddhism. This was in fact the main source of tensions in the country. In order to keep people calm, Tokugawa Ieyasu issued an edict ordering the Portuguese and Spanish to leave Japan, removal of missionaries, the destruction of all Christian churches, and forbidding Christianity in Japan. Tokugawa Ieyasu died a few years later, but his grandson Tokugawa Iemitsu finished the task of removing the foreigners. In 1641, there were practically no foreigners left in Japan.

All of these changes were the catalyst for new period in Japan, sometimes called sakoku, or a “Closed country”. Closing the country did not have exclusively negative effects. Most importantly, it stopped both internal and external conflicts. It also forced, and in fact helped, the Japanese to develop their own forms of art and science. The local art, science and culture started developing rapidly. This was also the case with mathematics. In this period Japanese mathematics (wasan) was born and developed.

It is difficult to say in what year exactly the tradition of sangaku began, but the oldest surviving sangaku dates from 1683 and was found in Togachi prefecture. Yamaguchi Kanzan, nineteenth century mathematician, mentions in his travel diary an even older tablet from 1668, but that one is now lost.

Over the next two centuries, the tablets spread and appeared all over Japan in Shinto shrines and Buddhist temples, two thirds of them in Shinto shrines. Many of the sangaku mentioned in contemporary mathematics texts were lost, but we can guess that there were originally thousands more than the 900 tablets which exist today. This practice of hanging tablets gradually died out after the fall of the Tokugawa shogunate, but some examples date as late as 1980. The latest sangaku were discovered in 2005. Five tablets were found in the Toyama prefecture. Earlier tablets were generally about 50 cm by 30 cm, but later tablets were sometimes as large as 180 cm by 90 cm, each displaying several geometry problems.

Some of the Japanese temple geometry (sangaku) or problems of Japanese mathematics before Meiji period (wasan) can be solved by using really useful method of inversion. There are many problems with multiple circles with a contact with one another. The main example was a problem proposed by Hotta Jinsuke and hung in 1788 at the Yanagijima Myōkendō temple of Tokyo. Yoshida Tameyuki solved this problem with traditional methods, this solution has been found in an unpublished manuscript “Solutions to Shinpeki Sanpō Problems". Yoshida's original solution of Hotta's problem was solved by using a Japanese equivalent of Descartes theorem, but this problem and many similar ones can be solved more easily by technique known as inversion. Inversion was discovered by western mathematicians between 1824 and 1845. This method was unknown to Japanese traditional mathematicians.

2. Hotta's problem and its traditional solution

As shown in Figure 2, a large circle of radius r contains two circles r_1 and r_1', each of radius $r_1 = \frac{r}{2}$, which are both tangent to each other and touch circle r internally. The bottom circle r_1' also touches a chain of cirlces r_n, as illustrated. Further, a chain of circles with radius t_n is placed between the circles r_n and r_1' such that each t_n touches r_1' as well as circles r_n and r_{n+1}. Find r_n and t_n in terms of n.

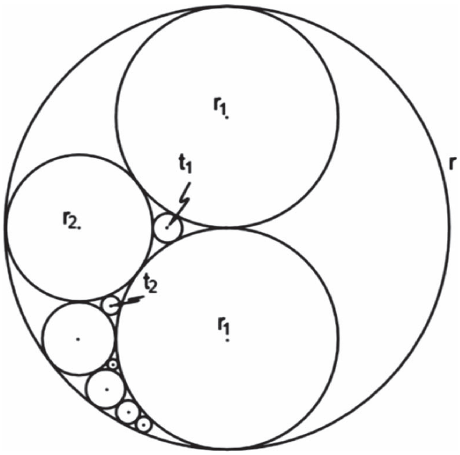

Figure 2

The Descartes circle theorem gives the relationship between the radii of four mutually tangent, or kissing, circles.

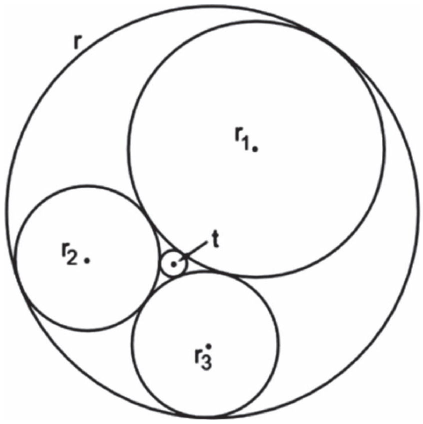

Figure 3

If three circles of radii r_1, r_2 and r_3 touch each other, touch a small circle of radius t externally and touch a circle of radius r internally, as shown in a Figure 3, than the following relation hold:

$$2\left(\frac{1}{r_1^{\,2}}+\frac{1}{r_2^{\,2}}+\frac{1}{r_3^{\,2}}+\frac{1}{t^2}\right)=\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{t}\right)^2,$$

$$2\left(\frac{1}{r_1^{\,2}}+\frac{1}{r_2^{\,2}}+\frac{1}{r_3^{\,2}}+\frac{1}{r^2}\right)=\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r}\right)^2.$$

Yoshida uses Descartes circle theorem on successive triplets of circles to inductively establish a recursion relationship for the r_n and t_n. For simplicity of the calculation, we will take $\frac{1}{r}=a$ and $\frac{1}{r_n}=p_n$, which we will be used in further calculation.

Let's find r_1 : $r_1=\frac{r}{2}$, $p_1=2a$.

Let's find r_2 : Using Descartes circle theorem for $\{r_1, r_1, r_2, r\}$, we get

$$2\left(\frac{1}{r_1^{\,2}}+\frac{1}{r_2^{\,2}}+\frac{1}{r_3^{\,2}}+\frac{1}{r^2}\right)=\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_2}-\frac{1}{r}\right)^2,$$

or

$$2\left(4a^2+4a^2+p_2^2+a^2\right)=\left(2a+2a+p_2-a\right)^2.$$

From above, we get $p_2 = 3a$ or $r_2 = \dfrac{r}{3}$.

Let's find r_3 : Using Descartes circle theorem for $\{r_1, r_2, r_3, r\}$, we get

$$2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r^2}\right)=\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r}\right)^2,$$

or

$$2\left(4a^2+9a^2+p_3^2+a^2\right)=\left(2a+3a+p_3-a\right)^2.$$

From above, we get $p_3 = 6a$ or $r_3 = \dfrac{r}{6}$.

Let's find r_4 : Using Descartes circle theorem for $\{r_1, r_3, r_4, r\}$,we get

$$2\left(\frac{1}{r_1^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}+\frac{1}{r^2}\right)=\left(\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_4}-\frac{1}{r}\right)^2,$$

or

$$82a^2+2p_4^2=\left(7a+p_4\right)^2.$$

From above, we get $p_4 = 11a$ or $r_4 = \dfrac{r}{11}$.

Let's find r_n : Using Descartes circle theorem for $\{r_1, r_n, r_{n+1}, r\}$, we get

$$2\left(p_1^2+p_n^2+p_{n+1}^2+a^2\right)=\left(p_1+p_n+p_{n+1}-a\right)^2,$$

or

$$p_{n+1}^2-2\left(a+p_n\right)p_{n+1}+10a^2+2p_n^2$$
$$-\left(a+p_n\right)^2=0.$$

Regarding this as a quadratic equation in $x = p_{n+1}$, the two solutions are $x_1 = p_{n+1}$, and $x_2 = p_{n-1}$. Then,

$$x_1+x_2=p_{n+1}+p_{n-1}=2\left(a+p_n\right),$$
or $\quad p_{n+1}-2p_n+p_{n-1}=2a$,

which is the desired recursion relationship.

The general solution:

$p_1 = 2a$, $p_2 = 3a = 2a + a$, $p_3 = 6a = 2a + 4a$,
$p_4 = 11a = 2a + 9a$, $p_5 = 18a = 2a + 16a \ \cdots$,
and $p_n = 2a + (n-1)^2 a$,

which yields $\quad r_n = \dfrac{r}{2+(n-1)^2}$.

To find t_n, Yoshida was using Descartes circle theorem for $\{r_n, r_{n+1}, t_n, r_1\}$. For simplicity, we will take q_n to be $q_n = \dfrac{1}{t_n}$,

$$2\left(p_n^2+p_{n+1}^2+q_n^2+p_1^2\right)=\left(p_n+p_{n+1}+q_n+p_1\right)^2.$$

Letting $p_n = 2a + (n-1)^2 a$ from above, we get quadratic equation in q_n,

$$q_n^2-2a\left(2n^2-2n+7\right)q_n$$
$$-\left(4n^2-4n+15\right)a^2=0,$$

$$\left\{q_n-\left(4n^2-4n+15\right)a\right\}\left\{q_n+a\right\}=0$$

Then we have solutions q_{n_1} and q_{n_2}, such that,

$$q_{n_1}=\left(4n^2-4n+15\right)a,\ q_{n_2}=-a.$$

We discard second solution and we get the final result

$$t_n=\frac{r}{(2n-1)^2+14},\ \text{or}\ n=\frac{1}{2}\left[\sqrt{\frac{r}{t_n}-14}+1\right],$$

which was written on the tablet.

Before we show a solution we get by using a method of inversion we have to introduce inversion.

3. Inversion

3.1 Definition of Inversion

Inversion is the process of transforming points P to a corresponding set of points P' known as their inverse points. Two points P and P' are said to be inverses with respect to an inversion circle having inversion center $T = (x_0, y_0)$ and inversion radius k if TP' is the perpendicular foot of the altitude of ΔTQP, where Q is a point on the circle such that TQ is perpendicualr to PQ.

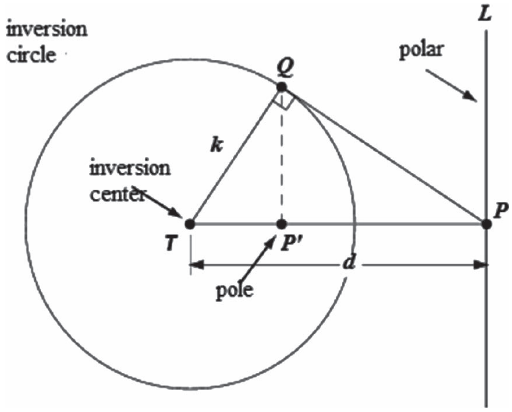

Figure 4

If P and P' are inverse points, then the line L through P and perpendicular to OP is sometimes called a "polar" with respect to point P', known as the "inversion pole". In addition, the curve to which a given curve is transformed under inversion is called its inverse curve or its inverse. From similar triangles, it immediately follows that the inverse points P and P' obey

$$\frac{TP}{k} = \frac{k}{TP'} \text{ or } k^2 = TP \cdot TP',$$

where the quantity k^2 is known as the circle power.

The general equation for the inverse of the point (x, y) relative to the inversion circle with the center of inversion (x_0, y_0) and inversion radius k is given by

$$x' = x_0 + \frac{k^2 (x - x_0)}{(x - x_0)^2 + (y - y_0)^2},$$

$$y' = y_0 + \frac{k^2 (y - y_0)}{(x - x_0)^2 + (y - y_0)^2}.$$

3.2 Properties of Inversions

In this section we are going to introduce a few Inversion theorems, some of which are going to be used in a proof of Hatta's theorem in chapter 4. We will give only the statements of theorems, without proofing them. The theorems will tell us how line and circles are going to be convert, relations between radius of original and converted circle and some other important relations.

Theorem 1.

A straight line passing through the center of inversion inverts into itself. A straight line not passing through the center of inversion inverts into a circle that passes through the center of inversion. (Figure 5)

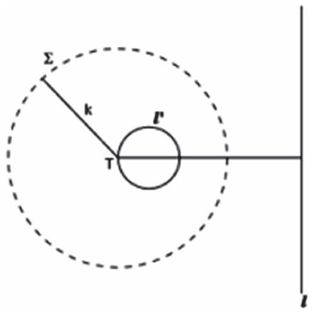

Figure 5

Theorem 2.

If circle C does not pass through the center of inversion T, then C inverts into another circle C'. (Figure 6)

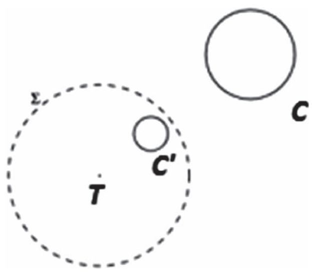

Figure 6

Theorem 3.

If circle C does pass through the center of inversion T, then C inverts into a straight line that does not pass through the center of the inversion. (Figure 7)

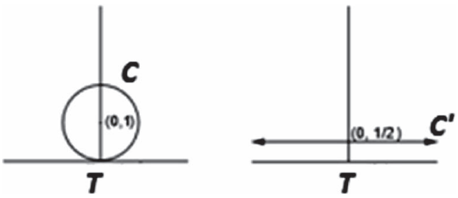

Figure 7

Theorem 4.

If r is the radius of C and r' is the radius of C', then r and r' are related by

$$r' = \frac{k^2}{\left|d^2 - r^2\right|} r,$$

where d is the distance between T and the center of C.

Theorem 5.

If L is the length of the tangent from T to the inverse circle C', then

$rL^2 = k^2 r'$. (Figure 8)

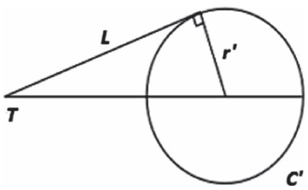

Figure 8

Theorem 6.

Point on the circle of inversion are invariant.

Theorem 7.

Concentric circles whose center is the center of inversion invert into concentric circles.

Theorem 8.

The center of the inverse circle is not the inverse of the center of the original circle.

Theorem 9.

If two circles are tangent to each other at T, they invert into parallel lines. If two circles are tangent to each other at a point P that is not the center of inversion, then the inverse circles must be tangent to each other at some point P'. Point of tangency are preserved.

Theorem 10.

A circle, it's inverse, and the center of inversion are collinear.

Theorem 11.

By the proper choice of the center of inversion T, two circles that are not in contact can be inverted into two concentric cirlces.

Theorem 12.

If four circles can be inverted into four circles of equal radii r', whose centers form the vertices of a rectangle, then

$$\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4},$$

where r_1, r_2, r_3, r_4 are the radii of the original circles.

4. Solution to Hatta's problem by using inversion

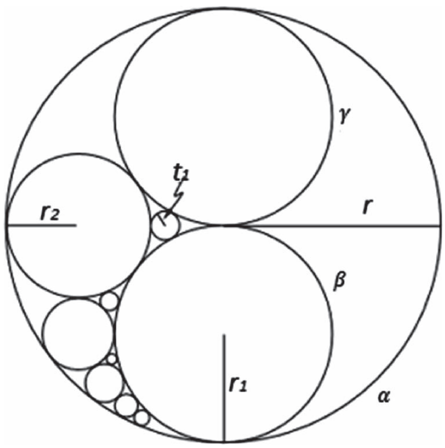

Figure 9

Radius of the outer circle α is r, radii of two largest inscribed circles β and γ are $r_1 = \frac{r}{2}$. We need to find and proof what is the radius of the nth circle in outer or inner contact chains in terms of r (r_n and t_n are used to designate the radii of the nth circle in the outer and inner chains, also to designate the circles themselves).

Pythagorean theorem gives $(r_1 + r_2)^2 = r_1^{\,2} + (r - r_2)^2$ which leads to $r_2 = \frac{1}{3}r$.

Similarly, by using Pythagorean theorem on small circle t_1 (Figure 10) one gets $t_1 = \frac{r}{15}$.

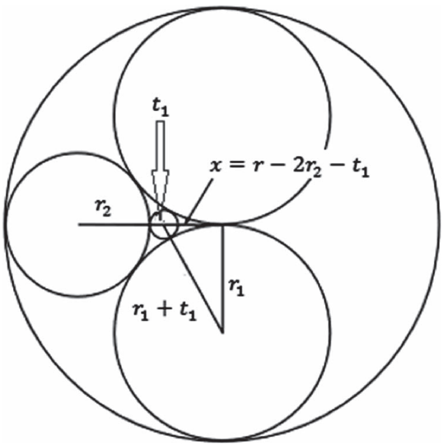

Figure 10

Now we'll start employing inversion, we'll invert figure with respect of the point T, chosen as shown in a Figure 11. Because they pass through the center of inversion T, α and β must invert into straight lines (Theorem 3), and horizontal line because we have chosen T to lie directly below O.

For simplicity, we will take the radius of inversion circle to be $k = 1$. By definition, we have $TO \cdot TO' = 1$. Then $TO = r$ so $TO' = \frac{1}{r}$. Similarly for point B, $TB \cdot TB' = 1$, $TB = 2r$ so $TB' = \frac{1}{2r}$.

Next we have to consider upper circle which does not pass trough T, so it must invert into another circle (Theorem 2). This circle is tangent

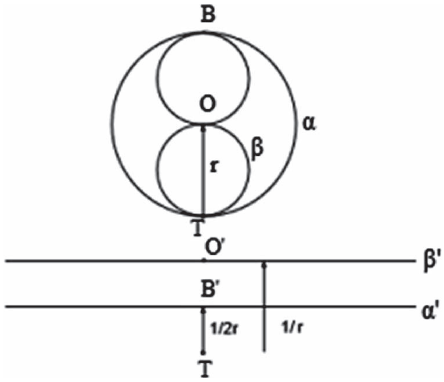

Figure 11

to to circles α and β so it must invert into a circle r_1' that lies between α' and β' as shown in a Figure 12.

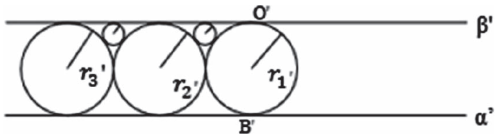

Figure 12

Similarly, circle r_2 is tangent to α, β and $\gamma = r_1$ so it must invert into the circle r_2' shown in above picture. The same is true for all the circles in outer chain. We get a result that all the inverse circles in outer chain have the same radius,

$$r_1' = r_2' = r_3' = \ldots = r_n' = r'.$$

In the same way we get that all the circles of inner chain invert into circles of equal radius,

$$t_1' = t_2' = t_3' = \ldots = t_n' = t'.$$

Let's relate r' and t' to r, considering $r_1 = \frac{r}{2}$.

The distance from T to a center of circle γ is d (by definition in Theorem 4), $d = 3r_1$. Theorem 4 states that $r_1'^2\left(d^2 - r_1^{\,2}\right)^2 = r_1^{\,2}$, which yields $r' = r_1' = \frac{1}{8r_1}$, $r' = \frac{1}{4r}$.
Similarly, $t' = \frac{1}{16r}$.

Now when we have r' and t' in terms of r

we can get r_n' and t_n'. Let L_n be a tangent from T to r_n', as show in a picture below.

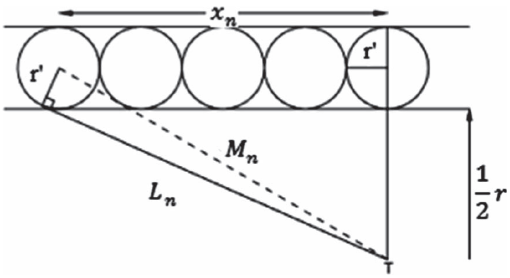

Figure 13

From the Figure 13 we can see that $x_n = 2(n-1)r'$. The distance L_n between T and r_n' can be calculated by Pythagorean theorem,

$$M_n^{\,2} = L_n^{\,2} + r'^2,\; M_n^{\,2} = \left(r' + \frac{1}{2r}\right)^2 + x_n^{\,2}.$$

By Theorem 5, $L_n^{\,2} = \dfrac{r'}{r_n}$.

By inserting r', x_n and L_n in above equation, one gets $r_n = \dfrac{r}{2+(n-1)^2}$.

For inner chain (Figure 14) procedure is similar,

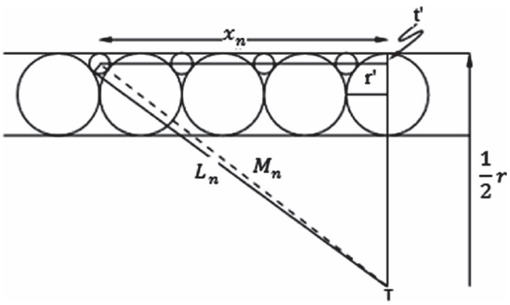

Figure 14

$$L_n^{\,2} = M_n^{\,2} - t'^2,\; M_n^{\,2} = \left(\frac{1}{r} - t'\right)^2 + x_n^{\,2}$$

In this case, Theorem 5 gives $L_n^{\,2} = \dfrac{t'}{t_n}$. By inserting t', x_n and L_n in above equation one gets,

$$t_n = \frac{r}{(2n-1)^2 + 14}.$$

Acknowledgment

Aljosa Matulic, one of the authors, has also learned why and how Japanese mathematics – Wasan was developed.[1] We have learned about birth, life and death of sangaku. Sangaku is at the same time a piece of art and a work of mathematics, but also a religious offering which was hanged in shrines and temples and that is why it is called “temple geometry”.

A. Matulic has found interesting the fact that Japanese mathematicians, in that period of closed country, had similar problems, which they’ve trying to proof and find solutions for, as their Western colleagues. Some areas of their interests were almost the same, they have just used different methods for solving the problems. Some of the sangaku problems are nice examples of that. Many of those problems were solved by using traditional Japanese methods.[2–6] If we try to solve some of these problems today, by using “western” theorems, one finds that solution can be much simpler but the final result is the same.

One of the methods which is really useful and ancient Japanese mathematicians didn’t know about if is method we introduced in this paper, method of inversion. This method makes a lot of problems understandable and easier to solve even to a high school students. Although is simple and useful, this method is not well known today because it isn’t taught in schools or universities any more (at least not in Croatian and many European schools and universities).

References

1) Fukagawa H. and Rothman T. (2008) Scared Mathematics, Japanese Temple Geometry. Princeton, New Jersey, USA; Woodstock, Oxfordshire, United Kingdom; Princeton University Press
2) Majewski M., Jen-Chung Chuan, Nishizawa H. The New Temple Geometry Problems in Hirotaka’s Ebisui Files, http://atcm.mathandtech.org/ep2010/invited/3052010_18118.pdf
3) Vincent J. and Vincent C. Japanese temple geometry, university of Melbourne, http://files.eric.ed.gov/fulltext/EJ720042.pdf
4) Murata T. Indigenous Japanese Mathematics, Wasan, http://fomalhautpsa.sakura.ne.jp/Science/Murata/Indigenous.pdf
5) Wolfram Math World, http://mathworld.wolfram.com/Inversion.html
6) Pinterest, https://www.pinterest.com/pin/546202261028328056/